metries become, so to speak, "quantized" by the self-consistency requirement, which determines not only the qualitative features of the allowable deviations from symmetry, but also their numerical magnitudes. In the model we have studied, in which eight vector mesons interact among themselves, self-consistency has led to a number of interesting results concerning the departure from SU_3 symmetry.

We found, first, that the model is very stable against a perturbation from symmetry which has the transformation properties of a 27-fold tensor, and much less stable against a perturbation of the 8-fold type.[18] This has the consequence that the model can be expected to have additional self-consistent solutions which have a small dissymmetry which is predominantly characterized by an 8-fold tensor, but does not have solutions with a small 27-fold dissymmetry. Since we consider a rather simplified model, and treat it only qualitatively, we do not attempt to calculate the numerical value of the dissymmetry. However, the fact that the magnitudes are determined by self-consistency leads at once, as we have shown, to retention of SU_2 symmetry. In other words, our model leads, in a naturalistic way, both to the Gell-Mann–Okubo mass formula and to the isotopic spin concept.

Finally, it should be pointed out that our present work is limited in three respects. First, we do not have a useful criterion for choosing between the completely symmetrical solution and the solution with perturbed symmetry; in fact, we have not even given an *a priori* reason for preferring SU_3 to any other group. Second, we have relied on qualitative arguments in estimating the parameters which describe the internal dynamical structure of the bound states. We should like to suggest, as a particularly useful program of numerical computation, the precise evaluation of bound state energies for a variety of input masses. This would determine these parameters more exactly, and also allow the exploration of the possibility of very unsymmetrical solutions to Eq. (1). Third, it is clear that the interrelations among the dissymmetries of different kinds of particles will be of particular interest. This last question we intend to discuss further in another paper.

[18] It would be quite wrong to speak of the symmetrical solution as being *unstable* against an 8-fold perturbation, since the magnitude of the deviation is, in fact, prescribed.

PHYSICAL REVIEW VOLUME 132, NUMBER 3 1 NOVEMBER 1963

Singularities of a Relativistic Scattering Amplitude in the Complex Angular Momentum Plane*

K. Bardakci and D. A. Geffen

School of Physics, University of Minnesota, Minneapolis, Minnesota

(Received 24 May 1963)

The analytic properties of the partial-wave amplitude in the complex angular momentum plane are investigated in a relativistic scalar-meson theory. Using a N/D decomposition, the numerator and denominator are calculated by perturbative expansions to fourth order in the coupling constant. Higher order poles at negative integer values of l are found in both the numerator and denominator, leading to a breakdown in their perturbation expansions near these singularities. The same breakdown occurs for all but the leading Regge trajectory. It is further found that, to fourth order, due to inelastic processes, the denominator has branch cuts for values of l near negative integers.

I. INTRODUCTION

We present here a relativistic model for scalar-meson–scalar-meson scattering and discuss the analytic properties of the resulting partial-wave scattering amplitude in the complex angular momentum plane. The method consists of decomposing the partial-wave amplitude, $a(s,l)$,

$$a(s,l)=N(s,l)[1+D(s,l)]^{-1}, \tag{1.1}$$

and calculating both N and D by perturbation expansions using an interaction of the form $(g/3!):\phi^3:$. The details of the method will be given in the next section.

Although the procedure is explicitly carried out to fourth order, we believe that some of the results established hold for all orders in g. One result is that the Regge trajectories

$$l=\alpha_n(s), \quad n=1, 2, \cdots, \tag{1.2}$$

with the possible exception of $n=1$, cannot be expanded in a power series in g. As we shall show later, this is intimately connected with the failure of the perturbation expansion near negative integer values of l.

Another result obtained is that the contribution of inelastic processes to the scattering amplitude leads, in a simple way, to the existence of branch cuts in the complex l plane for the denominator function to fourth

* Work supported in part by the U. S. Atomic Energy Commission.

order. These cuts are found to lie on the real l axis in the neighborhood of negative integers. Although the model we are using to obtain these results is clearly unphysical, we believe that the situation is very much the same in more realistic cases.

In Sec. II, we outline the general method for obtaining N and D in a perturbation expansion, and using the ϕ^3 interaction, apply the method to calculate N and D to fourth order in Secs. III through V. In these sections, we are primarily concerned with the singular behavior of N and D at negative integer values of l. Section VI is devoted to the existence of branch cuts due to inelastic processes. The last section contains a discussion of results, some conclusions and possible generalizations. All complicated expressions and lengthy derivations have been relegated to the Appendixes.

II. GENERAL METHOD

We assume that both N and D can be expanded in powers of g^2, where g is the coupling constant.

$$N(s,l)=\sum_{n=1}^{\infty} g^{2n}N^{(2n)}(s,l)\,,$$
$$D(s,l)=\sum_{n=1}^{\infty} g^{2n}D^{(2n)}(s,l)\,. \qquad (2.1)$$

To generate these series, assume that the scattering amplitude satisfies the Mandelstam representation,

$$f(s,t,u)=-\frac{g^2}{s-1}+\frac{1}{\pi}\int_4^\infty ds'\frac{\sigma(s')}{s'-s}+\frac{1}{\pi^2}\int_4^\infty\int_4^\infty ds'dt'\frac{\rho(s',t')}{(s'-s)(t'-t)}+\text{crossing symmetric terms}. \qquad (2.2)$$

We have taken the mass of the meson to be 1 and suppressed possible subtraction terms and convergence factors for the sake of simplicity.

The partial-wave amplitude $a(s,l)$ is given by

$$a(s,l)=\frac{4}{(s-4)\pi}\int_1^\infty dt\, Q_l\left(1+\frac{2t}{s-4}\right)A(s,t)\,, \qquad (2.3)$$

where[1]

$$A(s,t)=-g^2\pi\delta(t-1)+\frac{1}{\pi}\int_4^\infty ds'\frac{\rho(s',t)}{s'-s}+\frac{1}{\pi}\int_4^\infty du'\frac{\rho(u',t)}{t+s-4+u'}+\sigma(t)\,.$$

[1] The factor $\frac{1}{2}[1+e^{i\pi l}]$ multiplying $a(s,l)$ has been left out. This factor must be reintroduced in writing a Watson-Sommerfeld type representation.

The integral in Eq. (2.3) converges when the real part of l is sufficiently large and in this region defines an analytic function; further, it coincides with the physical partial waves at even integer values of l. The double dispersion relation implies that $a(s,l)$ is analytic in the complex s plane except for a cut running along the real axis.

One can expand $a(s,l)$ in powers of g^2,

$$a(s,l)=\sum_{n=1}^{\infty} g^{2n}a^{(2n)}(s,l)\,. \qquad (2.4)$$

Clearly, this expansion cannot converge when s and l approach a Regge trajectory. However, we expect that when $a(s,l)$ is written as a ratio of N and D, the perturbative expansions for N and D have a much larger domain of convergence. In fact, in nonrelativistic potential scattering, N and D have convergent expansions for all values of g^2, l, and s.[2]

The iteration procedure is begun by equating coefficients of g^2 on both sides of Eq. (1.1), which gives,

$$N^{(2)}(s,l)=a^{(2)}(s,l)\,. \qquad (2.5)$$

As usual, $N(s,l)$ is defined to have a branch cut only for $-\infty<s<4$, while D carries the other portion of the cut extending from $s=4$ to ∞, and satisfies the following dispersion relation:

$$D(s,l)=\frac{1}{\pi}\int_4^\infty ds'\frac{\Delta D(s',l)}{s'-s}\,, \qquad (2.6)$$

where $\Delta D(s',l)=(2i)^{-1}[D(s'+i\epsilon,\,l)-D(s'-i\epsilon,\,l)]$.

We obtain $\Delta D(s,l)$ from Eq. (1.1) for $s>4$,

$$N\Delta D=-[1+D_+][1+D_-]\Delta a\,, \qquad (2.7)$$

with $D_\pm=D(s\pm i\epsilon,\,l)$ and $\Delta a=(2i)^{-1}[a_+-a_-]$.

To find $\Delta D^{(2)}$, the right-hand side of (2.7) must be taken to fourth order. As we shall see later, for $s>4$, $\Delta a^{(2)}=0$ and,

$$\Delta a^{(4)}(s,l)=\rho(s)a_+^{(2)}(s,l)a_-^{(2)}(s,l)\,, \qquad (2.8)$$

where $\rho(s)=[(s-4)/s]^{1/2}$. Using these relations, we arrive at the well-known result,

$$\Delta D^{(2)}(s,l)=-\rho(s)N^{(2)}(s,l)\,. \qquad (2.9)$$

$N^{(4)}$ is given, by Eq. (1.1), in terms of the known quantities $a^{(4)}$, $a^{(2)}$, and $D^{(2)}$,

$$N^{(4)}(s,l)=a^{(4)}(s,l)+a^{(2)}(s,l)D^{(2)}(s,l)\,. \qquad (2.10)$$

Both terms on the right-hand side of Eq. (2.9) have cuts for $s>4$, but the discontinuities cancel by virtue of Eq. (2.8), so that $N^{(4)}(s,l)$ has a cut only for $s<4$.

$\Delta D^{(4)}$ can be calculated from Eq. (2.7),

$$-N^{(2)}\Delta D^{(4)}=N^{(4)}\Delta D^{(2)}+[D_+^{(2)}+D_-^{(2)}]\Delta a^{(4)}+\Delta a^{(6)}. \qquad (2.11)$$

[2] R. G. Newton, J. Math. Phys. 3, 867 (1962).

Since $\Delta a^{(6)}$ contains contributions from three-meson intermediate states, elastic unitarity no longer holds. However, we can write $a^{(6)}$ as a sum of elastic and inelastic terms,

$$a^{(6)}=a_e^{(6)}+a_i^{(6)}, \tag{2.12}$$

where $a_e^{(6)}$ consists of all elastic graphs and for $s>4$, satisfies

$$\Delta a_e^{(6)}=\rho[a_+^{(4)}+a_-^{(4)}]a^{(2)}. \tag{2.13}$$

Equations (2.12) and (2.13) enable us to write

$$\Delta D^{(4)}=-\rho N^{(4)}-(N^{(2)})^{-1}\Delta a_i^{(6)}$$

and

$$D^{(4)}(s,l)=-\frac{1}{\pi}\int_4^\infty ds'\,\rho(s')\frac{N^{(4)}(s',l)}{s'-s}-\frac{1}{\pi}\int_9^\infty ds'\frac{\Delta a_i^{(6)}(s',l)}{N^{(2)}(s',l)[s'-s]}. \tag{2.14}$$

The contribution from inelastic processes to $D^{(4)}$ in Eq. (2.14) is of special interest because of the presence of $N^{(2)}$ in the denominator of the integrand. This can give rise to branch cuts in the complex l plane if, for a range of values of l, $N^{(2)}(s,l)$ vanishes somewhere in the interval $9<s<\infty$. We shall see later that this does happen.

The iteration procedure outlined above can be carried out in a similar manner to yield N and D to arbitrary order. We shall not bother to write down the general expressions for N and D since we are not going to make any use of them.

At this point, it should be noted that the decomposition of the scattering amplitude made here is not unique even when we require that D contain the entire right-hand cut. It has been assumed that the denominator, $1+D$, contains no CDD poles and that it can be written in the form $1+\sum_{n=1}^\infty g^{2n}D^{(2n)}(s,l)$, rather than $P(s,g^2,l)+\sum_{n=1}^\infty g^{2n}D^{(2n)}(s,l)$, where $P(s,g^2,l)$ is an arbitrary polynomial in s. The only justifications for our particular choice of decomposition are its simplicity and analogy with nonrelativistic potential scattering. We should also point out that the dispersion relation for D in the case of ϕ^3 interaction needs no convergence factors, and we combine all possible additive polynomials in $P(s,g^2,l)$ mentioned above.

III. SECOND-ORDER TERMS IN N AND D

Since $\rho(s,t)$ and $\sigma(t)$ are at least of order g^4, $a^{(2)}(s,l)$ is obtained from Eq. (2.3) by substituting for $A(s,t)$ the single-meson exchange contribution $-g^2\pi\delta(t-1)$.

Using Eq. (2.5) we find,

$$N^{(2)}(s,l)=a^{(2)}(s,l)=-[4/(s-4)]\times Q_l[1+2/(s-4)]. \tag{3.1}$$

From the well-known properties of Q_l,[3] it follows that $N^{(2)}(s,l)$ is analytic in the entire complex l plane, except for the points $l=-n$, $n=1, 2, 3, \cdots$. At $l=-n$, $N^{(2)}(s,l)$ has a simple pole with residue

$$\gamma_n(s)=-[4/(s-4)]P_{n-1}[1+2/(s-4)]. \tag{3.2}$$

Given $N^{(2)}(s,l)$, we obtain $D^{(2)}(s,l)$ by means of Eqs. (2.6) and (2.9):

$$D^{(2)}(s,l)=\frac{4}{\pi}\int_1^\infty\frac{dz\,Q_l(z)}{(2z-1)^{1/2}[2+(4-s)(z-1)]}, \tag{3.3}$$

where we have introduced a simple change of variable.

Since $Q_l(z)$ behaves like z^{-l-1} as $z\to\infty$, the integral defining $D^{(2)}(s,l)$ converges for $\mathrm{Re}l>-\frac{3}{2}$ unless $s=4$. For $s=4$, the integral converges for $\mathrm{Re}l>-\frac{1}{2}$. Equation (3.3), therefore, defines $D^{(2)}(s,l)$ as an analytic function in the complex l plane for $\mathrm{Re}l>-\frac{1}{2}$ and in the entire complex s plane except for a branch cut along $4<s<\infty$.

The integral defining $D^{(2)}(s,l)$ can easily be continued into the entire complex l plane. To do so we need the analytic continuation of the function $I(w,l)$ defined by,

$$\begin{aligned}D^{(2)}(s,l)&=-\frac{4}{(s-4)\pi}I\left(1+\frac{2}{s-4},l\right),\\ I(w,l)&=\int_1^\infty\frac{dz\,Q_l(z)}{(2z-1)^{1/2}(z-w)},\end{aligned} \tag{3.4}$$

where w, in general, is complex. This can be accomplished by writing a dispersion relation for the function

$$F(z,l)=(2z-1)^{-1/2}Q_l(-z), \tag{3.5}$$

where the branch cut of $(2z-1)^{-1/2}$ is taken along $\frac{1}{2}<z<\infty$. Since $Q_l(-z)$ is an analytic function of z except for a branch cut along $-1<z<\infty$, $F(z,l)$ is an analytic function of z except for a branch cut extending from -1 to ∞. Furthermore, the discontinuity of F across the branch cut is known. It is easy to see from the properties of $Q_l(z)$ that the dispersion relation for F enables us to relate $I(w,l)$ to $F(w,l)$ and integrals over a finite range:

$$I(w,l)=\frac{\pi}{\cos\pi l}\left[\frac{-iQ_l(-w)}{[2w-1]^{1/2}}+\frac{1}{2}\int_{-1}^{1/2}\frac{P_l(-x)dx}{[1-2x]^{1/2}(x-w)}+\frac{1}{2\pi}\int_{1/2}^{1}dx\frac{Q_l(-x+i\epsilon)+Q_l(-x-i\epsilon)}{[2x-1]^{1/2}(x-w)}\right]. \tag{3.6}$$

The right-hand side of Eq. (3.6) defines $I(w,l)$ in the entire complex l plane except for simple poles for negative integral values of l, where Q_l has simple poles. Since this expression coincides with the original expression in Eq. (3.4) defining $I(w,l)$ only for $\mathrm{Re}l>-\frac{3}{2}$, Eq. (3.6) represents the analytic continuation of $I(w,l)$ into the entire complex l plane. Note that the vanishing of $\cos\pi l$ at half-integral values of l does not lead to

[3] A. Erdélyi, *Higher Trancendental Functions* (McGraw-Hill Book Company, Inc., New York, 1954), Vol. 1, Chap. 3.

poles in $I(w,l)$. This follows from the fact that $I(w,l)$ has no poles for positive half-integral values of l and the fact that $Q_l=Q_{-l-1}$ when l is a half-integer.

Equation (2.6) can be used to obtain the useful result, valid for $-\frac{3}{2}<\mathrm{Re}l<\frac{1}{2}$,

$$I(w,l)=-I(w,\,-l-1)+\frac{\pi}{\sin\pi l}\left[\int_{1/2}^{1}\frac{P_l(-x)dx}{(2x-1)^{1/2}(x-w)}-\frac{\pi i P_l(-w)}{(2w-1)^{1/2}}\right]. \quad (3.7)$$

The right-hand side of Eq. (3.7) then defines the analytic continuation of $I(w,l)$ for $\mathrm{Re}l<\frac{1}{2}$. There is no pole at $l=0$ since the residue vanishes identically.

Let us examine briefly the behavior of Regge poles both in the small-coupling and high-energy limits. The results we are going to get are not new, having been obtained by several authors,[4] and they are given here for the sake of completeness. Taking g^2 to be small, we restrict ourselves to the first-order contribution to D, obtaining,

$$1+g^2D^{(2)}(s,l)=0. \quad (3.8)$$

In the limit $g^2\to 0$ or $s\to\infty$, this equation can only be satisfied if l tends to a negative integer. If the trajectory that ends at $l=-n$ is $l=\alpha_n(s)$, the first-order contribution to α_n is given by

$$l=-n+\frac{4g^2}{(s-4)\pi}\left[\pi i\left(\frac{s-4}{s}\right)^{1/2}P_{n-1}\left(1+\frac{2}{s-4}\right)-\int_{1/2}^{1}dx\frac{P_{n-1}(x)}{(2x-1)^{1/2}\left(x-1-\dfrac{2}{s-4}\right)}\right]=\alpha_n(s). \quad (3.9)$$

As $s\to\infty$, the leading term in $\alpha_n(s)$ is,

$$\alpha_n(s)\to -n+[4g^2/(4-s)\pi]\ln(4-s). \quad (3.10)$$

We now can draw the following conclusions about Regge trajectories: As s starts from a large negative value and moves to the right on the real axis in the s plane, $\alpha_n(s)$ starts from $-n$ and moves to the right in the l plane. It is clear that $\alpha_n(s)$ cannot reach the value $-n+1$ because of the singularity of $D^{(2)}$ at this point; therefore, all trajectories except for α_1 must become complex for some real value of s. It can be shown that this happens for $s<4$, and that all the trajectories end at $l=-\frac{1}{2}$ when s reaches 4. Since $\alpha_n{}^*$ is also a solution to Eq. (3.8), the trajectories appear in complex conjugate pairs. Only the leading trajectory $\alpha_1(s)$ remains real and reaches a value $l>-\frac{1}{2}$ at $s=4$. For $s>4$, $\mathrm{Im}s>0$, the α_n become complex with $\mathrm{Im}\alpha_n>0$ until they turn back and approach the negative integer points as $s\to+\infty$.

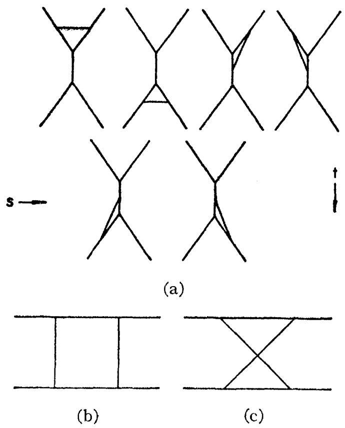

FIG. 1. (a) Radiative corrections to the vertex function; (b) direct box diagram; (c) crossed box diagram.

The similarity of these results with those obtained in potential scattering theory is obvious.[2] Unfortunately, Eq. (3.9) is no longer valid if the fourth-order contribution to D is taken into account, as we shall see later. However, we may hope that some of the qualitative results still remain valid.

IV. FOURTH-ORDER CONTRIBUTIONS TO THE NUMERATOR

$N^{(4)}$, the fourth-order contribution to the numerator, is given in terms of $a^{(4)}$ by Eq. (2.10). $a^{(4)}$ in turn is determined from $A^{(4)}(s\,t)$ through Eq. (2.3). The fourth-order graphs that contribute to the absorbtive function are shown in Figs. 1(a) (b), and (c). The diagrams in Fig. 1(a) are the first-order radiative corrections to the vertex function, and their total contribution is denoted by $A_v{}^{(4)}(s,t)$; Figs. 1(b) and 1(c) give the direct and crossed box graphs, and their contributions are labeled $A_d{}^{(4)}$ and $A_c{}^{(4)}$, respectively.

Denoting the corresponding partial-wave amplitudes by $a_v{}^{(4)}$, $a_d{}^{(4)}$, and $a_c{}^{(4)}$, we have the following explicit expressions:

$$\begin{aligned}
a_v{}^{(4)}(s,l)&=\frac{3}{\pi^2(s-4)}\int_4^\infty dt\,Q_l\left(1+\frac{2t}{s-4}\right)\frac{\ln(t-3)}{(t-1)[t(t-4)]^{1/2}},\\
a_d{}^{(4)}(s,l)&=-\frac{1}{2\pi^2(s-4)}\int_4^\infty dt\,Q_l\left(1+\frac{2t}{s-4}\right)\int_4^\infty\frac{ds'}{s'-s}\frac{\theta((s'-4)(t-4)-4)}{[s't[(s'-4)(t-4)-4]]^{1/2}},\\
a_c{}^{(4)}(s,l)&=-\frac{1}{2\pi^2(s-4)}\int_4^\infty dt\,Q_l\left(1+\frac{2t}{s-4}\right)\int_4^\infty ds'\frac{\theta((s'-4)(t-4)-4)}{(s'+s+t-4)[s't[(s'-4)(t-4)-4]]^{1/2}}.
\end{aligned} \quad (4.1)$$

[4] See, for example, B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266 (1962).

We note that, because of the unitarity condition for the box diagram, $a_d^{(4)}$ satisfies the elastic unitarity relation $\Delta a_d^{(4)}=\rho[a^{(2)}]^2$ for $s>4$.

Since $Q_l(z)$ behaves like z^{-l-1} for large Z, the integrals for $a_v^{(4)}$, $a_d^{(4)}$, and $a_c^{(4)}$ in Eq. (4.1) converge for $\mathrm{Re}(l)>-2$, $\mathrm{Re}(l)>-1$, and $\mathrm{Re}(l)>-2$, respectively. These integrals can be continued to the left-half complex l plane by means of a procedure similar to that used for $D^{(2)}$ in Sec. III. For example, the result for $a_v^{(4)}$ is

$$a_v^{(4)}(s,l)=\frac{3}{\pi(s-4)\sin(\pi l)}\left\{\frac{i}{\sqrt{3}}\left[\ln(-2)+\frac{\pi e^{-i\pi l}}{\sin(\pi l)}\right]Q_l\left(-1-\frac{2}{s-4}\right)+\tfrac{1}{2}(s-4)\int_0^1 dt\frac{P_l(2t-1)}{(4-s)t-1}\right.$$
$$\times\frac{\ln[(4-s)t-3]+\pi e^{-i\pi l}/\sin(\pi l)}{[t(4-s)[(4-s)t-4]]^{1/2}}+\frac{i\cos(\pi l)}{\pi}\int_0^4 dtQ_l\left(1+\frac{2t}{s-4}\right)\frac{\ln(t-3)+(\pi e^{-i\pi l})/\sin(\pi l)}{(t-1)[t(t-4)]^{1/2}}$$
$$\left.-e^{-i\pi l}\int_3^4 dt\,Q_l[1+2t/(s-4)]/[(t-1)(t(t-4))^{1/2}]\right\}. \quad (4.2)$$

The expressions for $a_d^{(4)}$ and $a_c^{(4)}$ are more complicated, and together with their derivations, they are given in Appendix A.

Equation (4.2) defines a function analytic in the product of the cut s plane and the entire l plane with the exception of negative integer points. At these points there are, in general, third-order poles due to the terms of the form $[\sin(\pi l)]^{-2}Q_l$. At the point $l=-1$, however, the original expression given in Eq. (4.1) is valid, and the pole at this point is a simple one. Since $a^{(4)}=a_v^{(4)}+a_d^{(4)}+a_c^{(4)}$, we have also to consider the terms $a_d^{(4)}$ and $a_c^{(4)}$. From the expressions given in Appendix A, it follows that $a_c^{(4)}$ has a simple pole at $l=-1$ and third-order poles at every negative odd integer starting with $l=-3$, whereas $a_d^{(4)}$ has double poles at all negative integers, including $l=-1$. Therefore, $a^{(4)}$ must have higher order poles at negative integer values of l (in general third order), and a complete cancellation of these poles between different terms, in general, cannot occur. The same conclusion also applies to $N^{(4)}$, since the additional term $a^{(2)}D^{(2)}$ in Eq. (2.10) has only second-order poles.

These results have some bearing on the asymptotic behavior of individual terms in a perturbation expansion. If a bounded function $f(s,t)$ has a partial-wave amplitude $a(s,l)$ which is analytic in the entire l plane except for poles at negative integer points, then it admits the following asymptotic expansion in inverse powers of t,

$$f(s,t)=\sum_{n=1}^{N}c_n(s)t^{-n-1/2}+\sum_{n=1}^{N}t^{-n}\sum_{m=0}^{i_n-1}d_{nm}(s)[\ln(t)]^m+R_N(s,t). \quad (4.3)$$

In this equation, N is a positive integer, i_n in the order of the pole at $l=-n$, and the background term $R_N(s,t)$ goes like t^{-N-1} as $t\to\infty$. In the first sum, the coefficient $c_n(s)$ is proportional to $[a(s,n-\frac{1}{2})-a(s,-n-\frac{1}{2})]$ and vanishes if $a(s,n-\frac{1}{2})=a(s,-n-\frac{1}{2})$.

As a result of this theorem, it follows that the fourth-order terms in the perturbation series have asymptotic expansion given in Eq. (4.3), which is a generalization of the results of Federbush and Grisaru.[5] The leading terms for $a_d^{(4)}$, $a_c^{(4)}$, and $a_v^{(4)}$ are of the form $\ln(t)/t$, $1/t$, and $1/t$, respectively. It is also interesting to note that the first sum in Eq. (4.3) drops out in the case of $a_d^{(4)}$.[6]

Equation (4.3) is a simple generalization of the result derived by Mandelstam[7] for the nonrelativistic scattering amplitude. In the case he treats, the first sum involving half-integer powers of t drops out; also the poles he considers are Regge poles and not the static poles that arise here. Of course, no Regge poles occur in the direct perturbative expansion of the amplitude. However, his arguments can be taken over with minor modifications to establish Eq. (4.3).

Another important consequence of the results of this section concerns the validity of the perturbation expansion of N. We have shown that $N^{(4)}$ has higher order poles than $N^{(2)}$, and it seems very likely that the order of the poles at negative integer points keep on increasing with the order of the perturbation expansion. This, of course, means that the perturbation expansion must fail near these points. The failure of perturbation theory at negative integer values of l supports the conjecture that the partial-wave amplitude must have essential singularities at these points.[8]

It is of some interest to see what happens if one considers a Bethe–Salpeter-type amplitude which does not have crossing symmetry. The perturbation expansion of such an amplitude does not have terms of the form $a_c^{(4)}$ and $a_v^{(4)}$, and to the fourth order, only $a_d^{(4)}$ contributes. In this case, we have $N^{(4)}=a_d^{(4)}+N^{(2)}D^{(2)}$, and it is shown in Appendix B that the second-order poles of $a_d^{(4)}$ are not, in general, canceled by the corresponding poles of $a^{(2)}D^{(2)}$, except at $l=-1$. There-

[5] P. Federbush and M. Grisaru (to be published).

[6] This follows from the fact that $a_d^{(4)}$ satisfies the condition $a_d^{(4)}(s,n-\frac{1}{2})=a_d^{(4)}(s,-n-\frac{1}{2})$. For details, see Appendix A.

[7] S. Mandelstam, Ann. Phys. (N. Y.) **19**, 1254 (1962).

[8] V. N. Gribov and I. Ya. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. **43**, 308 (1962) [translation: Soviet Phys.—JETP **16**, 220 (1963)].

fore, it seems very likely that the perturbation expansion of N fails even for the Bethe-Salpeter amplitude if $\mathrm{Re}\,l \leqq -2$. We shall see in the next section that similar conclusions apply also to D. The latter result has special significance to the discussion of the $g^2=0$ limit of Regge trajectories.

V. FOURTH-ORDER CONTRIBUTIONS TO THE DENOMINATOR

The starting point is Eq. (2.14). We separate $D^{(4)}$ into an elastic part $D_e^{(4)}$ and an inelastic part $D_i^{(4)}$,

$$D^{(4)}=D_e^{(4)}+D_i^{(4)}\,,$$
$$D_e^{(4)}=-\frac{1}{\pi}\int_4^\infty ds'\,\rho(s')\frac{N^{(4)}(s',l)}{s'-s}\,, \tag{5.1}$$
$$D_i^{(4)}=-\frac{1}{\pi}\int_g^\infty ds'\frac{\Delta a_i^{(6)}(s',l)}{(s'-s)N^{(2)}(s',l)}\,.$$

Let us first examine the convergence properties of the integral for $D_e^{(4)}$. Since $N^{(4)}(s,l)$ goes to zero as $s\to\infty$, this integral converges at the upper limit. Near $s=4$, however, $N^{(4)}$ behaves like $(s-4)^l$, and, therefore, the integral in question diverges for $\mathrm{Re}(l)\leqq -1$. (The properties of $N^{(4)}$ required in the above discussion are derived in Appendix A.) An analytic continuation for $D_e^{(4)}$ is provided by the following formula:

$$D_e^{(4)}(s,l)=-\frac{i\rho(s)}{\cos(\pi l)}e^{\mp i\pi l}N^{(4)}(s,l)$$
$$+\frac{1}{2\pi\cos(\pi l)}\int_{-\infty}^0\frac{ds'}{s'-s}\rho(s'+i\epsilon)[e^{-i\pi l}N^{(4)}(s'+i\epsilon,\,l)$$
$$+e^{i\pi l}N^{(4)}(s'-i\epsilon,\,l)]\,, \tag{5.2}$$

where the sign of the term $(i\pi l)$ in the exponential is opposite to the sign of $\mathrm{Im}(s)$, and the kinematical factor $\rho(s)=[(s-4)/s]^{1/2}$ is taken to have cuts running from 0 to $-\infty$ and from 4 to ∞. To prove Eq. (5.2), we note that its right-hand side defines an analytic function with a cut on the real axis. It is easily seen that the jump across this cut for $s<0$ vanishes identically. One can also verify from Eqs. (2.3) and (4.1) that,

$$e^{-i\pi l}N(s+i\epsilon,\,l)-e^{i\pi l}N(s-i\epsilon,\,l)=0\,, \tag{5.3}$$

for $0<s<4$. It follows from this equation that the jump of $D_e^{(4)}$ is defined by Eq. (5.2) between $s=0$, and $s=4$ vanishes. Finally, the jump of $D_e^{(4)}$ is equal to $-2i\rho N^{(4)}(s,l)$ when $s>4$. Equation (5.2) is thus seen to be completely equivalent to Eq. (5.1).

The integral in Eq. (5.2) converges for all values of l since the threshold point $s=4$ is not included in the range of integration. It then follows that $D_e^{(4)}(s,l)$ is analytic in the entire complex l plane except at negative integer points, where it has in general the same order poles as $N^{(4)}(s,l)$. These poles, with the exception of the pole at $l=-1$, are, in general, of higher order, as was shown in the previous section, and they make a simple perturbative expansion for the Regge trajectories except for the leading one impossible. To see this, consider the limit $g^2\to 0$ of the equation

$$1+g^2D^{(2)}+g^4D_e^{(4)}=0\,. \tag{5.4}$$

Clearly, in this limit, Eq. (5.4) can only be satisfied if l approaches the singularities of D as $g^2\to 0$. This result was obtained to second order in Sec. III and led us to define the trajectories as $l=\alpha_n(s)$, $n=1,\,2,\,\cdots$, $l\to -n$ as $g^2\to 0$. But $D_e^{(4)}$ has third-order poles at all negative integer values of l except $l=-1$, where the pole is first order. Consequently, in the neighborhood of $l=-n$, $n\neq 1$, Eq. (5.4) takes the form

$$1+g^2R^{(2)}(s)/(l+n)+g^4R_e^{(4)}(s)/(l+n)^3=0\,. \tag{5.5}$$

A solution to (5.5) as $g^2\to 0$ requires that $\alpha_n\to -n+O(g^{4/3})$. This result is completely inconsistent with a perturbative expansion of α_n in powers of g^2. On the other hand, the leading trajectory, $l=\alpha_1(s)$ has no such difficulties (to fourth order at least).

Of course, it is possible that the third (and second!) order poles in $D_e^{(4)}$ are canceled out by contributions from $D^{(4)}$ and higher order terms in the expansion for D. Quite the contrary is expected, namely, the order of the poles of $D^{(2n)}$ and $N^{(2n)}$ at negative integral values of l should increase rapidly with n. Of course, this result is closely connected with the fact that perturbation theory for D itself is not valid.

Let us consider, now, the inelastic part of $D^{(4)}$, $D_i^{(4)}$, defined in Eq. (5.1). The integral as it stands is defined for real values of $l>-1$. It is shown, however, in Appendix D, that $\Delta a_i^{(6)}(s,l)$ is an analytic function of l in the entire complex plane except for poles at negative integer points. Since $N^{(2)}(s,l)$ has similar properties in the complex variable l, $\Delta D_i^{(4)}(s,l)$ is analytic in the entire l plane with the exception of negative integer points and at values of l (for fixed s) where $N^{(2)}(s,l)$ vanishes. The analytic continuation for $D_i^{(4)}$ itself can then be obtained directly from the analytic continuation of $\Delta a_i^{(6)}/N^{(2)}$.[9] $D_i^{(4)}$ is not defined for a given value of l if, in the range of integration of s', $N^{(2)}(s',l)$ has a zero.

From Eq. (3.1), we see that $N^{(2)}(s',l)$ vanishes at the zeros of $Q_l(1+2/(s'-4))$. Since s' varies from 9 to ∞, the argument of Q_l varies from 7/5 to 1. We show in Appendix C that, as z varies from 1 to ∞, the values of l for which $Q_l(z)$ vanish vary continuously from $-n$ to $-n-\frac{1}{2}$ along the real l axis. Therefore, the function $D_i^{(4)}(s,l)$, and, hence, $D(s,l)$ has branch cuts which extend to the left along the real l axis, beginning at

[9] $\Delta a_i^{(6)}(s,l)$ may be singular at the point $s=9$ for $\mathrm{Re}(l)<0$. This difficulty can be circumvented by writing the integral for $D_i^{(4)}$ as a contour integral around the right-hand cut of $a_i^{(6)}(s',l)/(s'-s)N^{(2)}(s',l)$. Since the integrand is analytic in s', we can deform the contour slightly in the s' plane to avoid the point $s'=9$. This does not change any of our conclusions.

negative integers and ending before reaching the neighboring negative half-integer. [We ignore here the possibility that $\Delta a_i^{(6)}(s',l)$ vanishes when $N^{(2)}(s',l)$ does, since this seems highly unlikely.] It must not be concluded, from the existence of branch cuts in the denominator function to fourth order, that the scattering amplitude itself, given by $N/(1+D)$, has these cuts as well. We reserve the discussion of this point to the conclusions that follow.

VI. CONCLUSIONS

We have investigated the complex angular momentum properties of the scattering amplitude using a N/D factorization to obtain perturbation expansions for both N and D for the case of scalar mesons with a ϕ^3 interaction. Since our calculations are only up to fourth order in the coupling constant, the validity of our conclusions depend upon the unknown convergence properties of the series expansions for both the numerator and denominator functions. Consequently, our results are of only suggestive value. While we expect that these expansions remain valid in a larger region of the l plane than does the usual series for $a(s,l)$ given by Eq. (2.4), we can only hope that this region is large enough to include at least the leading Regge trajectory. It is further assumed that the analytic continuation of N and D to the left of $\mathrm{Re}(l)=-1$ is given by the sums of the analytic continuations of each term in their perturbative expansions. From our results to fourth order that higher order terms in the series contribute poles of increasing order at $l=-2, -3, \cdots$, it is very likely that the expansions actually diverge in the neighborhood of these points. This has the immediate consequence that the Regge trajectories, $l=\alpha_n(s,g^2)$, with the possible exception of the leading one, are not expandible in powers of g^2. Hence, one must solve the equation $1+D(s,l)=0$ without resort to a perturbative expansion in order to determine these trajectories.

The factorization of the $a(s,l)$ made in this paper requires that N carry the whole left-hand cut in s and D carry the entire right-hand cut. Several other plausible methods of factorization exist; for example, N may be required to have part of the inelastic right-hand cut. However, if analogy with the Fredholm solution to the potential scattering problem is maintained, no ambiguity arises in the treatment of elastic graphs (i.e., graphs with only two-particle intermediate lines in the s channel). When one takes into account inelastic processes, the situation changes, and different methods of factorization yield different numerator and denominator functions.

With our choice of factorization, we have found that D, to fourth order, contains cuts in the complex l plane near negative integer points. We cannot conclude from this, however, that these cuts are also present in the scattering amplitude obtained after summing the series for N and D. Indeed, it is easy to find other factorizations which do not lead to cuts in the l plane for individual terms in the perturbative expansions. This does not preclude the possibility that the sum has branch cuts nor does it necessarily follow that an arbitrary choice of factorizations will yield convergent series for N and D. In our case, it is easily seen that the numerator function also has similar cuts starting with the sixth-order term in the expansion. Hence, if the power series in g^2 for the numerator and denominator functions converge absolutely in a region of the plane including the cut, it can be shown that their ratio is free of any cuts. In the absence of any definite knowledge of the validity of the particular factorization we have used, as well as of the convergence properties of the resulting power series, our results only suggest that inelastic processes can give rise to branch cuts in the complex l plane near negative integer points. Note that the cuts in the l plane we have found arise in quite a different manner from cuts obtained previously by Sawyer[10] and Swift and Lee.[11]

The nature of the higher order poles at negative integer values of l supports the conjecture of Gribov and Pomeranchuk[8] about the existence of essential singularities at these points. The above-mentioned authors reached this conclusion using crossing symmetry; however, from our analysis in Sec. IV, it seems likely the situation is the same for the Bethe-Salpeter amplitude. It is also remarkable that, at least to the order g^4, there is no trouble at $l=-1$. This leaves some hopes for the expandability of the leading Regge trajectory in powers of g^2.

The methods employed so far for the scalar theory can also be applied to more realistic theories such as pseudoscalar mesons interacting through a ϕ^4 term. Because of the more singular high-energy behavior of the interaction compared to the ϕ^3 interaction, difficulties arise to the left of $\mathrm{Re}(l)=0$ instead of $\mathrm{Re}(l)=-1$. Again, there is indication of breakdown of perturbation theory at negative integer points, and branch cuts arise due to inelastic processes. The location of these branch cuts, however, seems to be a difficult task, and a result considerably different from the ϕ^3 case is possible.

ACKNOWLEDGMENTS

The authors would like to express their gratitude to S. Gasiorowicz for his contributions and valuable criticisms during the progress of this work. One of the authors acknowledges a helpful conversation with M. L. Goldberger and F. E. Low.

[10] R. F. Sawyer (to be published).

[11] A. R. Swift and B. W. Lee (to be published).

APPENDIX A: ANALYTIC CONTINUATION OF FOURTH-ORDER GRAPHS

We present here explicit formulas for the analytic continuations of $a_d^{(4)}$ and $a_c^{(4)}$,

$$a_d^{(4)}(s,l)=-\frac{1}{2\pi(s-4)\sin(\pi l)}\Bigg\{\int_4^\infty ds'\frac{1}{s'-s}\Bigg[\frac{i\cos(\pi l)}{\pi}\int_0^4 dt\frac{Q_l[1+2t/(s-4)]}{\{s't[(s'-4)(t-4)-4]\}^{1/2}}$$
$$-\frac{1}{2}\int_0^1 dt\frac{P_l(2t-1)}{\left[s't\left(s't-4\left(\frac{s'}{4-s}+t\right)+\frac{12}{4-s}\right)\right]^{1/2}}\Bigg]-\int_0^4\frac{dv}{(v-4)^{1/2}}\Bigg[\frac{Q_l\left(-1-\frac{2v}{(s-4)^2}-\frac{8}{s-4}\right)}{(v+4s-16)^{1/2}(s(s-4))^{1/2}}$$
$$+\frac{1}{2}\int_0^1 dt\frac{P_l(2t-1)[(4-s)t-4]^{1/2}}{[(4-s)t-v/(s-4)-4]^{1/2}\{(4-s)t[(4-s)t+v-16]\}^{1/2}}$$
$$+\frac{1}{\pi}\int_0^4 dt\frac{(t-4)^{1/2}Q_l(1+2t/(s-4))}{[(t-4)(s-4)-v][t(4t+v-16)]^{1/2}}\left[\theta\left(t-4+\frac{v}{4}\right)\sin(\pi l)-i\cos(\pi l)\theta\left(4-\frac{v}{4}-t\right)\right]\Bigg]\Bigg\}. \quad \text{(A1)}$$

$$a_c^{(4)}(s,l)=-\frac{1}{4\pi(s-4)\sin(\pi l)}\Bigg\{\frac{1}{2}\int_4^\infty ds'\int_0^1 dt\frac{(s-4)P_l(2t-1)}{[(4-s)(t-1)+s']\{s't(4-s)[(4-s)s't-4(s'+(4-s)t)+12]\}^{1/2}}$$
$$-\frac{i(s-4)}{\sin(\pi l)}\int_0^1 dt\,P_l(2t-1)\int_0^1 dz\Bigg[\frac{2[z(z-1)((s-4)(2-Z)+Z\kappa(s))]^{1/2}[(s-4)(1-z)+(z+1)\kappa(s)]^{1/2}}{2(4-s)t-Z(4-s+\kappa(s))}$$
$$-\frac{2[z(z-1)((s-4)(z-2)-zK(s))]^{1/2}[(s-4)(z-1)-(z+1)K(s)]^{1/2}}{2(4-s)(t-1)-Z(s-4-\kappa(s))}\Bigg]$$
$$+\frac{2i}{\sin(\pi l)}\int_0^1 dZ\Bigg[\frac{Q_l\left(z-1-\frac{zK(s)}{s-4}\right)}{[z(z-1)((s-4)(2-z)+zK(s))]^{1/2}[(s-4)(1-z)+(z+1)\kappa(s)]^{1/2}}$$
$$-\frac{Q_l\left(1-z+z\frac{\kappa(s)}{s-4}\right)}{[z(z-1)((s-4)(z-2)-zK(s))]^{1/2}[(s-4)(Z-1)-(Z+1)\kappa(s)]^{1/2}}\Bigg]$$
$$+\int_0^4 ds'\frac{Q_l\left(1+\frac{2s'}{s-4}\right)}{[s'(s'+s-4)(s'^2+s'(s-4)+4(1-s))]^{1/2}}+\frac{i\cos(\pi l)}{\pi}\int_4^\infty\frac{ds'}{s'-s}$$
$$\times\int_0^4 dt\frac{Q_l(1+2t/(s-4))}{(t+s+s'-4)[s't((s'-4)(t-4)-4)]^{1/2}}+\int_0^4\frac{dv}{(v-4)^{1/2}}$$
$$\times\Bigg[-\frac{i\pi}{\tau(s,v)\cos(\pi l)}\Bigg\{\frac{[4+s+\tau(s,v)]^{1/2}}{[4-s-\tau(s,v)]^{1/2}}\frac{Q_l\left(-\frac{\tau(s,v)}{s-4}\right)}{[8+2s+2\tau(s,v)-v]^{1/2}}-\left[\frac{4+s-\tau(s,v)}{4-s+\tau(s,v)}\right]^{1/2}$$
$$\times\frac{Q_l\left(\frac{\tau(s,v)}{s-4}\right)}{[8+2s+2\tau(s,v)-v]^{1/2}}\Bigg\}+\frac{i\pi(s-4)}{2\cos(\pi l)}\int_0^1 dt\frac{[(4-s)t-4]^{1/2}P_l(2t-1)}{[(4-s)^2(t^2-1)+v-4s][(4-s)t((4-s)t+v-16)]^{1/2}}$$

$$-\int_0^{4-v/4} dt\frac{(t-4)^{1/2}Q_l(1+2t/(s-4))}{[t^2+(s-4)t+v-4s][t(4t+v-16)]^{1/2}}$$

$$+i\tan(\pi l)\int_{4-v/4}^{4} dt\frac{(t-4)^{1/2}Q_l(1+2t/(s-4))}{[t^2+(s-4)t+v-4s][t(4t+v-16)]^{1/2}}\Bigg]\Bigg\}. \quad \text{(A2)}$$

We shall now sketch the derivations of Eqs. (4.2), (A1), and (A2). Consider the function

$$I_v(x,s,l)=\left[\ln(x-3)+\frac{\pi e^{-i\pi l}}{\sin(\pi l)}\right][x(x-4)]^{1/2}Q_l\left(-1-\frac{2x}{s-4}\right),$$

which has a cut from $x=4-s$ to $s=\infty$ for real s. The discontinuities of this function across various parts of its branch cut can be calculated through the use of the following identities:

$$Q_l(-x\mp i\epsilon)=-e^{\pm i\pi l}Q_l(x\pm i\epsilon),\quad \sin(\pi l)Q_l(x\pm i\epsilon)=\tfrac{1}{2}\pi[e^{\mp i\pi l}P_l(x)-P_l(-x)], \quad \text{(A3)}$$

with x real. It can then be verified that a dispersion relation for $I_v(x,s,l)$ yields Eq. (4.2).

To derive Eqs. (A1) and (A2), we convert Eqs. (4.1) in the following form:

$$a_d^{(4)}(s,l)=-\frac{1}{2\pi(s-4)\sin(\pi l)}\int_4^\infty\frac{ds'}{s'-s}\left\{\frac{1}{2}\int_{4-s}^0 dt\frac{P_l[-1-2t/(s-4)]}{[s't((s'-4)(t-4)-4)]^{1/2}}\right.$$
$$\left.+\frac{i\cos(\pi l)}{\pi}\int_0^{(4s'-12)/(s'-4)}dt\frac{Q_l[1+2t/(s-4)]}{[s't((s'-4)(t-4)-4)]^{1/2}}\right\},$$

$$a_c^{(4)}(s,l)=-\frac{1}{2\pi(s-4)\sin(\pi l)}\int_4^\infty ds'\left\{\frac{Q_l[1+2s'/(s-4)]}{[s'(s'+s-4)(s'^2+s'(s-4)+4(1-s))]^{1/2}}\right.$$
$$+\int_{4-s}^0 dt\frac{P_l[-1-2t/(s-4)]}{(t+s+s'-4)[s't((s'-4)(t-4)-4)]^{1/2}}$$
$$\left.+\frac{i\cos(\pi l)}{\pi}\int_0^{(4s'-12)/(s'-4)}dt\frac{Q_l[1+2t/(s-4)]}{(t+s+s'-4)[s't((s'-4)(t-4)-4)]^{1/2}}\right\}. \quad \text{(A4)}$$

The above equations can easily be proved by considering the dispersion relation the function

$$Q_l[-1-2x/(s-4)]/[x^{1/2}((s'-4)(x-4)-4)]^{1/2}$$

satisfies.

In the expression for $a_d^{(4)}$, only the last integral needs a further analytic continuation. To this end, we first simplify the integral by the substitution $(s'-4)(t-4)=v$, and then transform it into an expression that does not involve infinite limits of integration through the use of the dispersion relation the function $[(x-4)/x(v+4x-16)]^{1/2}Q_l[-1-2x/(s-4)]$ satisfies. As for the expression for $a_c^{(4)}$ in Eq. (A4), the first integral that occurs in it can be transformed into a suitable form by the standard trick of considering a dispersion relation for the integrand. The same can be done for the last integral after a change of variable $(s'-4)(t-4)=v$, which completes the derivation.

It is clear from Eq. (A4) that, in general, $a_d^{(4)}$ has second order and $a_c^{(4)}$ has third-order poles at negative integer points due to the factors of $\sin(\pi l)$ in the denominator and Q_l in the numerator. The pole at $l=-1$ in $a_c^{(4)}$ and $a_v^{(4)}$ is, however, simple, since the original expressions for them given in Eq. (4.1) converge at that point.

There are some points of interest in connection with the expressions given in Eqs. (A1) and (A2). It is clear from these expressions that both $a_d^{(4)}(s,l)$ and $a_c^{(4)}(s,l)$, as well as $a_v^{(4)}(s,l)$, go to zero as $s\to\infty$ even when $\mathrm{Re}(l)<0$. For $\mathrm{Re}(l)<0$, $a_c^{(4)}$ and $a_v^{(4)}$ both behave as $(s-4)^l$ near $s=4$, whereas $a_d^{(4)}$ behaves as $(s-4)^{2l}$. We shall see in the next Appendix that this spurious threshold contribution gets cancelled by the term $a^{(2)}D^{(2)}$. Another important point to note is the fact that the factor $[\cos(\pi l)]^{-1}$ that occurs in the expression for $a_c^{(4)}$ does not give rise to poles at half-integer points, since the expression that multiplies this term vanishes at the same points. Finally, $a_d^{(4)}$ satisfies the relation $a_d^{(4)}(s,n-\frac{1}{2})=a_d^{(4)}(s,-n-\frac{1}{2})$ when n is an integer, since all the terms in the expression for $a_d^{(4)}$

satisfy it individually. The same conclusion does not hold for $a_c^{(4)}$, however, since there is a $\cos(\pi l)$ in the denominator that vanishes at $l=n-\frac{1}{2}$, and the indefinite expression when evaluated properly can be shown to violate this reflection principle. Hence, we see that in contradistinction to potential scattering, the relativistic amplitude in general cannot satisfy the equation $a(s, n-\frac{1}{2})=a(s, -n-\frac{1}{2})$.

APPENDIX B: THE DIRECT TERMS IN THE NUMERATOR

Here we investigate the possible cancellations between the terms $a_d^{(4)}$ and $a^{(2)}D^{(2)}$ that contribute to $N^{(4)}$. Denoting $N_{\text{B.S.}}^{(4)}(s,l)=a_d^{(4)}(s,l)+a^{(2)}D^{(2)}$, where $N_{\text{B.S.}}^{(4)}$ is the Bethe-Salpeter numerator to the fourth order, it is easy to see that $N_{\text{B.S.}}^{(4)}$ has no right-hand cut in the s plane because of the unitarity relation $\Delta a_d^{(4)}=\rho[a^{(2)}]^2$. Therefore, $(s-4)^{-l}\ N_{\text{B.S.}}^{(4)}$ has no singularities to the right of $s=4$, and $N^{(4)}$ must behave like $(s-4)^l$ near $s=4$, although the terms $a_d^{(4)}$ and $a^{(2)}D^{(2)}$ individually have the spurious threshold behavior mentioned in Appendix A.

We now show that the second-order pole of $N^{(4)}$ at $l=-n$ has a residue $\sigma_n(s)$ given by

$$\sigma_n(s)=T_n[1/(s-4)], \tag{B1}$$

where T_n is an nth order polynomial. The following formula can easily be derived from Eqs. (2.3) and (4.1):

$$e^{-i\pi l}N_{\text{B.S.}}^{(4)}(s+i\epsilon, l)-e^{i\pi l}N_{\text{B.S.}}^{(4)}(s-i\epsilon, l) =\frac{4i\theta(-s)}{s-4}\int_4^{4-s}dt\, P_l\left(-1-\frac{2t}{s-4}\right)A_d^{(4)}(s,t) +\frac{4i}{s-4}\theta(3-s)P_l\left(-1-\frac{2}{s-4}\right)D^{(2)}(s,l). \tag{B2}$$

In deriving Eq. (B2), use is made of the fact that $A_d^{(4)}(s,t)$ has no left-hand cut in the s plane. The integral that appears in this equation is an entire function of l because of the finite range of integration, and the last term has only first-order poles due to the term $D^{(2)}$. Taking l to be a negative integer, it follows that the residue of a second-order pole $\sigma_n(s)$ cannot have a left-hand cut. Further, it can have no right-hand cut since $N_{\text{B.S.}}^{(4)}$ has none. Using the threshold behavior of $N_{\text{B.S.}}^{(4)}$, we conclude that $\sigma_n(s)$ must be an nth order polynomial in $1/(s-4)$. An explicit calculation using Eqs. (3.7) and (A1) shows that this polynomial vanishes for $n=1$, but does not in general vanish for other values of n. We therefore conclude that $N_{\text{B.S.}}^{(4)}$ has, in general, second-order poles at negative integers with the exception of $l=-1$. Here it is of some interest to remark that the kinematical factor $\rho(s)$ is entirely responsible for the second-order poles in the relativistic case; in the corresponding situation in potential scattering the second-order poles of $a_d^{(4)}$ and $a^{(2)}D^{(2)}$ cancel at every negative integer point.

APPENDIX C: THE ZEROS OF $Q_l(z)$

Since the vanishing of $Q_l(z)$ for real values of z greater than 1 lead, in our model, to branch cuts in the complex l plane, we present here some properties of the zeros of $Q_l(z)$ that are not readily available in the literature.

We observe, first, that the function $\zeta^{l+1}Q_l(z)$, where $\zeta=z+(z^2-1)^{1/2}$, is bounded in the complex l plane for $|z|>1$ except for the presence of simple poles at $l=-n-1$, $n=0, 1, \cdots$. Consequently, we can write

$$\zeta^{l+1}Q_l(z)=\sum_{n=0}^{\infty}\frac{P_n(z)}{\zeta^n(l+n+1)}. \tag{C1}$$

If we take the imaginary part of Eq. (C1) we obtain, immediately, the result

$$Q_l(z)\neq 0 \quad \text{for} \quad \operatorname{Im} l\neq 0,\quad z>1 \quad \text{and real}. \tag{C2}$$

For l real, $Q_l(z)>0$ for real $z>1$ and $l>-1$. This can be deduced from Eq. (C1), since every term in the sum is positive. Finally we note that, for fixed $z>1$, $Q_l(z)$ changes sign as l varies continuously from $-n-\epsilon$ to $-n-1+\epsilon$, $n=1, 2, \cdots$. Since $Q_l(z)$ is a continuous function of l in these intervals, it follows that $Q_l(z)=0$ for some value of l in the region $-n-1<l<-n$, $n=1, 2, \cdots$.

The positions of the roots of $Q_l(z)$ as z varies from 1 to ∞ can be given more specifically. To do so we make use of the known behavior of $Q_l(z)$ for Z near 1 and for $z\gg 1$. It is then easy to prove that the roots of $Q_l(z)=0$ satisfy the following properties:

$$\begin{aligned}&\text{As} \quad z\to 1+,\quad l+n\to 0-,\\ &\text{for} \quad z\to\infty,\quad l+n+\tfrac{1}{2}\to 0+,\quad n=1, 2, \cdots.\end{aligned} \tag{C3}$$

Since $Q_l(z)$ cannot vanish at the points $l=-n, -n-\frac{1}{2}$, $n=1, 2, \cdots$, it follows that the roots of $Q_l(z)=0$ are confined to the intervals $-n-\frac{1}{2}<l<-n$, $n=1, 2, \cdots$, as z varies from 1 to ∞.

We summarize the results of Appendix C in the following theorem:

The equation $Q_l(z)=0$, with $\operatorname{Im} z=0$ and $1<z<\infty$ can only be satisfied for real values of l which are restricted to lie in the intervals

$$-n-\tfrac{1}{2}<l<-n,\quad n=1, 2, \cdots.$$

As z approaches 1 in the interval, l approaches $-n$, while it reaches the value $-n-\frac{1}{2}$ only when z becomes infinite.

APPENDIX D: ANALYTICITY PROPERTIES OF SIXTH-ORDER GRAPHS

Here we want to examine the analyticity region of the inelastic part of $a^{(6)}$. Since we want a nonvanishing $\Delta a^{(6)}$ for $s>9$, we have to consider graphs with three-particle cuts in the s channel. We shall restrict ourselves

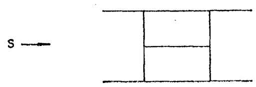

FIG. 2. A typical sixth-order diagram.

to the graph given in Fig. 2, the treatment of other graphs being similar. If the contribution of this graph to the scattering amplitude is denoted by $f_1^{(6)}(s,t)$, by standard Feynman parametrization, we have

$$f_1^{(6)}(s,t)=\frac{g^6}{(2\pi)^8}\int_0^1 d\alpha\delta[1-\sum\alpha_i]\Lambda(\alpha)$$
$$\times[A(\alpha)s+B(\alpha)t+C(\alpha)+i\epsilon]^{-3}, \quad \text{(D1)}$$

where the integration runs over seven Feynman parameters $\alpha_1, \cdots, \alpha_7$ symbolically denoted by a single letter α, and,

$$A(\alpha)=\alpha_1\alpha_2\alpha_3,$$
$$B(\alpha)=\alpha_7\alpha_6[\alpha_1+\alpha_2+\alpha_4+\alpha_5]+\alpha_4\alpha_5[\alpha_2+\alpha_3+\alpha_6+\alpha_7]+\alpha_2(\alpha_7\alpha_4+\alpha_6\alpha_5),$$
$$C(\alpha)=\alpha_2^2(\alpha_1+\alpha_2+\alpha_3)-(\alpha_1+\alpha_2+\alpha_4+\alpha_5)\times[(\alpha_1+\alpha_2+\alpha_3)(\alpha_2+\alpha_3+\alpha_6+\alpha_7)+(\alpha_6+\alpha_7)^2]-(\alpha_4+\alpha_5)[2\alpha_2(\alpha_6+\alpha_7)+(\alpha_4+\alpha_5)(\alpha_2+\alpha_3+\alpha_6+\alpha_7)],$$
$$\Lambda(\alpha)=(1-\alpha_2)(\alpha_2+\alpha_3+\alpha_6+\alpha_7)-(\alpha_3+\alpha_6+\alpha_7)^2.$$

We choose the physical t channel so that $t>4$ and $s<0$ and sufficiently small in magnitude. The absortive function in this channel, given by $A_1^{(6)}(s,t)=[f(s,t)-f^x(s,t)]/2i$, is equal to

$$A_1^{(6)}(s,t)=\frac{g^6}{4(2\pi)^7}\int_0^1 d\alpha\delta[1-\sum\alpha_i]\Lambda(\alpha)$$
$$\times\delta''[A(\alpha)s+B(\alpha)t+C(\alpha)]. \quad \text{(D2)}$$

With the help of Eq. (2.3), the corresponding partial amplitude $a_1^{(6)}(s,l)$ is given by

$$a_1^{(6)}(s,l)=\frac{8g^6}{(s-4)^3(2\pi)^8}\int_0^1 d\alpha\delta(1-\sum\alpha_i)\frac{\Lambda}{[B(\alpha)]^3}$$
$$\times Q_l''\left(1-\frac{2A(\alpha)}{B(\alpha)}-\frac{2}{s-4}\frac{4A(\alpha)+C(\alpha)}{B(\alpha)}\right). \quad \text{(D3)}$$

Although Eq. (D3) has so far been established only for restricted values of s, it can easily be continued to the entire complex s plane except for the usual cuts on the real axis. This follows by observing that the function Q_l'' has a cut only on the real axis, and that the imaginary part of the expression $Z=1-2A(\alpha)/B(\alpha)-[2/(s-4)]4A(\alpha)+C(\alpha)/B(\alpha)$ has the opposite sign compared to Im(s), since the function $(4A+C)/B$ is negative semidefinite.

We want to extend Eq. (D3) to values of l in the left-half complex l plane. As it stands, this expression blows up for $\mathrm{Re}(l)<-1$, whenever $B(\alpha)$ vanishes. To get around this difficulty, we use the following series expansion for the function $Q_l''(Z)$ in inverse powers of Z:

$$Q_l''(z)=\pi^{1/2}2^{-l+1}\frac{\Gamma(l+1)}{\Gamma(\frac{1}{2}l+\frac{1}{2})\Gamma(\frac{1}{2}l+1)}z^{-l-3}$$
$$\times\sum_{n=0}^{N-1}z^{-2n}\frac{\Gamma(\frac{1}{2}l+2+n)\Gamma(\frac{1}{2}l+\frac{3}{2}+n)}{n!\Gamma(l+\frac{3}{2}+n)}+K_N(z,l), \quad \text{(D4)}$$

where N is an arbitrary positive integer and the remainder term $K_N(z,l)$ goes at least as fast as $|z|^{-2N-l-3}$ as $z\to\infty$. Substituting Eq. (D4) into Eq. (D3), we get

$$a_1^{(6)}(s,l)=\frac{8g^6}{(s-4)^3(2\pi)^8}\pi^{1/2}2^{-l+1}\frac{\Gamma(l+1)}{\Gamma(\frac{1}{2}l+\frac{1}{2})\Gamma(\frac{1}{2}l+1)}$$
$$\times\sum_{n=0}^{N}\frac{\Gamma(\frac{1}{2}l+2+n)\Gamma(\frac{1}{2}l+\frac{3}{2}+n)}{n!\Gamma(l+\frac{3}{2}+n)}W_n(s,l)$$
$$+\bar{K}_N(s,l), \quad \text{(D5)}$$

where

$$\bar{K}_N(s,l)=\int_C^1 d\alpha\,\delta(1-\sum\alpha_i)[B(\alpha)]^{-3}\Lambda(\alpha)\times K_N\left(1-\frac{2}{s-4}\frac{A(\alpha)s+C(\alpha)}{B(\alpha)}\right),$$

$$W_n(s,l)=\int_0^1 d\alpha\,\delta[1-\sum\alpha_i][B(\alpha)]^{l+2n}\Lambda(\alpha)\times\left\{B(\alpha)-\frac{2}{s-4}[A(\alpha)s+C(\alpha)]\right\}^{-l-3-2n}.$$

The integral for $\bar{K}_N(s,l)$ is well defined except when $B(\alpha)$ becomes zero, at which point the argument of K_N becomes infinite. Since $K_N(z,l)$ goes as $|z|^{-2N-l-3}$ as $|z|\to\infty$, it is easily seen that the integral for K_N converges for $\mathrm{Re}(l)>-2N-1$ and defines an analytic function in this region. Let us now examine the term $W_n(s,l)$. It can be easily shown that in the range of integration of α's, the expression $Z(\alpha)=B(\alpha)-2[A(\alpha)s+C(\alpha)]/(s-4)$ can never vanish if $\mathrm{Im}(s)\neq0$. Then taking Im$s\neq0$, the integral for $W_n(s,l)$ is well defined except when $B(\alpha)=0$. $B(\alpha)$ can vanish only if some of the α's vanish, and for the sake of definiteness we take $\alpha_4=\alpha_7=0$, the argument being similar for other pairs of α's. Defining

$$C'(\alpha_1,\alpha_2,\alpha_3,\alpha_5,\alpha_6)\equiv C(\alpha_1,\alpha_2,\alpha_3,0,\alpha_5,\alpha_6,0),$$
$$Z'(\alpha_1,\alpha_2,\alpha_3,\alpha_5,\alpha_6,s)\equiv Z(\alpha_1,\alpha_2,\alpha_3,0,\alpha_5,\alpha_6,0)=\alpha_2\alpha_6\alpha_5-[2/(s-4)]\times[\alpha_1\alpha_2\alpha_3s+C'(\alpha)],$$
$$\Lambda'(\alpha)=(1-\alpha_2)(\alpha_2+\alpha_3+\alpha_6)-(\alpha_3+\alpha_6)^2,$$

we can write[12]

[12] To be more rigorous we should have defined W_n^{II} with α_4

$$W_n(s,l)=W_n{}^{\mathrm{I}}(s,l)+W_n{}^{\mathrm{II}}(s,l)\,,$$

where

$$W_n{}^{\mathrm{I}}(s,l)=\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_5 d\alpha_6 \delta[1-\sum \alpha_i] \times\Lambda'(\alpha)(\alpha_2\alpha_5\alpha_6)^{l+2n}[Z'(a,s)]^{-l-3-2n}, \quad \text{(D6)}$$

and $W_n{}^{\mathrm{II}}$ is regular at $\alpha_4=\alpha_7=0$. The integral for $W_n{}^{\mathrm{I}}$ diverges when α_2, α_5, or α_6 is near zero unless $\mathrm{Re}(l)>-2n-1$. To increase the domain of convergence in the l plane, we integrate by parts with respect to α_2, α_5, and α_6, successively, and obtain the following result:

$$\begin{aligned}
W_n{}^{\mathrm{I}}(s,l)=&\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_5 d\alpha_6\, \delta[1-\sum \alpha_i]\frac{(\alpha_2\alpha_5\alpha_6)^{l+2n+1}}{(l+2n+1)^3}\bigg\{\bigg[\bigg(\delta(\alpha_3)+\frac{\partial}{\partial\alpha_3}-\frac{\partial}{\partial\alpha_6}\bigg)\bigg(\frac{\partial}{\partial\alpha_3}-\frac{\partial}{\partial\alpha_5}\bigg)\bigg(\delta(\alpha_1)+\frac{\partial}{\partial\alpha_1}-\frac{\partial}{\partial\alpha_2}\bigg)\\
&+\bigg(\delta(\alpha_1)+\frac{\partial}{\partial\alpha_1}-\frac{\partial}{\partial\alpha_6}\bigg)\delta(\alpha_3)\bigg(\frac{\partial}{\partial\alpha_1}-\frac{\partial}{\partial\alpha_2}\bigg)\bigg][\Lambda'(\alpha)Z'(\alpha,s)^{-l-2n-3}]\bigg\}+\int_{\epsilon_1}^1 d\alpha_6\int_0^1 d\alpha_5\int_0^1 d\alpha_2\delta(1-\alpha_2-\alpha_5-\alpha_6)\\
&\times\frac{(\alpha_2\alpha_5)^{l+2n+1}}{(l+2n+1)^2}(\alpha_6)^{l+2n}\Lambda'(\alpha)Z'(\alpha,s)^{-l-2n-3}+\int_0^{\epsilon_1}d\alpha_6\int_0^{\epsilon_2}d\alpha_5\frac{[\alpha_5\alpha_6(1-\alpha_5-\alpha_6)]^{l+2n+1}}{(l+2n+1)^3}\\
&\times\bigg\{\bigg[\delta(\alpha_6-\epsilon_1)-\frac{\partial}{\partial\alpha_6}\bigg]\bigg[\frac{(1-\alpha_5-\alpha_6)[(1-\alpha_2)(\alpha_2+\alpha_6)-\alpha_6{}^2]}{1-\alpha_5-2\alpha_6}Z'(0,\,1-\alpha_5-\alpha_6,\,0,\,\alpha_5,\,\alpha_6,\,s)^{-l-2n-3}\bigg]\bigg\}\\
&-\int_0^1 d\alpha_2\int_0^{1-\alpha_2}d\alpha_6\frac{[\alpha_2\alpha_6(1-\alpha_2-\alpha_6)]^{l+2n+1}}{(l+2n+1)^3}\frac{\partial}{\partial\alpha_6}\bigg\{\theta(1-\alpha_2-\alpha_6-\epsilon_2)\theta(\epsilon_1+\alpha_2-1)\frac{1-\alpha_2-\alpha_6}{1-\alpha_2-2\alpha_6}\\
&\qquad\times[(1-\alpha_2)(\alpha_2+\alpha_6)-\alpha_6{}^2]Z'(0,\,\alpha_2,\,0,\,1-\alpha_2-\alpha_6,\,\alpha_6,\,s)^{-l-2n-3}\bigg\}. \qquad \text{(D7)}
\end{aligned}$$

In the above expression, the arbitrary constants ϵ_1 and ϵ_2 satisfy the relations $\epsilon_1>0$, $\epsilon_2>\epsilon_1$, and $\epsilon_1+2\epsilon_2<1$, so that the expressions $(1-\alpha_5-2\alpha_6)^{-1}$ and $(1-\alpha_2-2\alpha_6)^{-1}$ that occur in some of the terms never blow up. Indeed this is the reason for splitting up the range of integration with respect to α_2 and α_5 into intervals $(0,\epsilon_1)$, $(\epsilon_1,1)$, and $(0,\epsilon_2)$, $(\epsilon_2,1)$ in the double-surface term one gets after two partial integrations; the last partial integration is carried out with respect to α_5 or α_2, depending on the interval.

Inspecting the exponent of the expression $(\alpha_2\alpha_5\alpha_6)$ and the corresponding intervals of integration in various terms in Eq. (D7), we see that domain convergence has been increased from the region $\mathrm{Re}(l)>-2n-2$ to $\mathrm{Re}(l)>-2n-3$. This process of partial integration can be repeated indefinitely, and it follows that $W_n{}^{\mathrm{I}}$, and, therefore, W_n is analytic in the complex l plane excluding negative integer points. Using Eq. (D5), it is clear that $a_1{}^{(6)}(s,l)$ is analytic for $\mathrm{Re}(l)>-2N-1$, except for the usual poles. Since N can be taken arbitrarily large, $a_1{}^{(6)}(s,l)$ must be meromorphic in the entire plane, concluding the argument.

The preceding discussion has many points of similarity to the treatment of the asymptotic properties of perturbation terms given in Ref. 5. Indeed, the discussion in Sec. IV shows that the analyticity properties proved above imply the asymptotic expansion given by Eq. (4.3) for $a_1{}^{(6)}(s,l)$.

We finally remark that, although we have here chosen a specific graph for illustration, the ideas of the proof apply to any arbitrary diagram.

---

and α_7 restricted to a small region near zero, rather than actually setting $\alpha_4=\alpha_7=0$. The argument that follows is not, however, substantially effected by this simplification. We thank Dr. Schnitzer and Dr. Federbush for bringing this point to our attention.